

Generalized spherically symmetric gravitational model: Hamiltonian dynamics in extended phase space and BRST charge

T. P. Shestakova

Department of Theoretical and Computational Physics, Southern Federal University,

Sorge St. 5, Rostov-on-Don 344090, Russia

E-mail: shestakova@sfedu.ru

Abstract

We construct Hamiltonian dynamics of the generalized spherically symmetric gravitational model in extended phase space. We start from the Faddeev – Popov effective action with gauge-fixing and ghost terms, making use of gauge conditions in differential form. It enables us to introduce missing velocities into the Lagrangian and then construct a Hamiltonian function according a usual rule which is applied for systems without constraints. The main feature of Hamiltonian dynamics in extended phase space is that it can be proved to be completely equivalent to Lagrangian dynamics derived from the effective action. The sets of Lagrangian and Hamiltonian equations are not gauge invariant in general. We demonstrate that solutions to the obtained equations include those of the gauge invariant Einstein equations, and also discuss a possible role of gauge-noninvariant terms. Then, we find a BRST invariant form of the effective action by adding terms not affecting Lagrangian equations. After all, we construct the BRST charge according to the Noether theorem. Our algorithm differs from that by Batalin, Fradkin and Vilkovisky, but the resulting BRST charge generates correct transformations for all gravitational degrees of freedom including gauge ones. Generalized spherically symmetric model imitates the full gravitational theory much better than models with finite number of degrees of freedom, so that one can expect appropriate results in the case of the full theory.

1. Introduction

In [1] we present the outline of Hamiltonian dynamics in extended phase space that can be constructed to be completely equivalent to a Lagrangian formulation for a system with constraints. As a rule, Hamiltonian formulation for gravity is constructed following to the Dirac scheme [2, 3] and is a starting point for most attempts to quantize gravity. However, there are some reasons to doubt that Dirac Hamiltonian formulation for gravitational theory [4] can be thought as an equivalent one to the original General Relativity. The reasons are closely connected with the role that given to gauge gravitational degrees of freedom $g_{0\mu}$, $\mu = 0, 1, 2, 3$, fixing a reference frame, in these two formulations.

In Einstein (Lagrangian) formulation of General Relativity $g_{0\mu}$ components of metric tensor are treated on an equal basis with the rest of components, g_{ij} , $i, j = 1, 2, 3$, defining 3-space geometry. The theory is invariant under gauge transformations, an infinitesimal form of which is

$$\delta g_{\mu\nu} = \eta^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \eta^\lambda + g_{\nu\lambda} \partial_\mu \eta^\lambda \quad (1.1)$$

and which are true for all components of metric tensor (η^μ are infinitesimal parameters). In the Dirac approach only g_{ij} components with their conjugate momenta are included into phase space, the transformations for this variables are generated by constraints. The constraints, which do not depend on gauge variables, play a crucial role in quantization procedure, while gauge variables are dropped out of further consideration.

It is well-known that the Wheeler – DeWitt quantum geometrodynamics [5], constructed by a direct application of the Dirac quantization scheme to gravity, has faced quite a number of problems (see, for example, [6, 7]). At the same time, modern approaches to quantization of gravity, such as Loop Quantum Gravity, keep considering the Wheeler – DeWitt equation as a main constituent of the theory [8].

It is worth mentioning that the algebra of constraints in original phase space not including gauge degrees of freedom does depend on parametrization of gravitational variables. As was argued in our previous paper [1], it creates a serious obstacle to find an algorithm to construct a generator that would give correct transformations for all gravitational variables in the limits of the Dirac approach. Moreover, any transformations of gravitational variables, which touch upon gauge degrees of freedom, are not canonical from the viewpoint of the Dirac formalism. Even a transition from metric tensor components to the Arnowitt – Deser – Misner (ADM) variables [9], quite legitimate in the Lagrangian formulation, should be considered as non-canonical [10]. In this case the theory turns out to be essentially dependent on a chosen parametrization.

In the framework of the canonical scheme it is not obvious that the original (Lagrangian) formulation of General Relativity and its Hamiltonian dynamics à la Dirac are theories with different groups of transformations. It has become more clear in the Batalin – Fradkin – Vilkovisky (BFV) approach [11, 12, 13], that aims at reproducing the Dirac results on the path integral level. A Hamiltonian form of the effective action in this approach is determined by algebra of constraints. The algebra of constraints in the case of gravity is open and, as was emphasized in [11], gauge transformations differ from transformations generated by gravitational constraints. It may lead to a new type of additional Feynman diagrams corresponding to four-ghosts interaction which cannot result from the effective action in the Lagrangian form. The new type of diagram does not play an important role while one is interested in only gauge invariant sector in the S -matrix theory, for which the BFV approach was originally proposed. However, we can think of it as a considerable mathematical indication that Lagrangian and Hamiltonian formalism appear to be non-equivalent for the full theory of gravity when one deals with spacetime manifolds of any topology, in particular, without asymptotic states which are implied in the S -matrix approach.

The central part in the BFV approach is given to the BRST charge constructed as a series in powers of Grassmannian variables with coefficients given by generalized structure functions of constraints algebra [14]. Like the constraints, the BRST charge does not generate correct gauge transformations for all gravitational degrees of freedom including gauge ones. It is not surprising because the form of the BRST charge is determined by constraints algebra and, as was already mentioned, gauge transformations differ from those generated by the constraints.

The purpose of this paper is to present Hamiltonian dynamics in extended phase space which is free from

the shortcomings mentioned above and can be thought of as a real alternative for Dirac generalized Hamiltonian dynamics, as well for the BFV formalism. The proposed approach has already been demonstrated for gravitational models with finite degrees of freedom ([15, 16, 1] and other papers). In this work we apply our approach to generalized spherically symmetric gravitational model which imitates the full gravitational theory much better, so that one can see the way how one can get appropriate results in the case of the full theory. In Section 2 the Lagrangian and Hamiltonian dynamics in extended phase space are derived from the effective action for the model, and the structure of the Hamiltonian function and Hamiltonian equations is analyzed. In Section 3 we shall consider some particular cases that the generalized model embraces, and show that the solutions to the obtained equations comprise solutions to the gauge invariant Einstein equations with corresponding symmetry. We also discuss a possible role of gauge-noninvariant terms. In Section 4 we shall make use of BRST invariance of effective action and construct the BRST charge according to the Noether theorem which generates correct transformations for all gravitational degrees of freedom. The proposed method will be shown to be self-consistent, and the equivalence of the Lagrangian and Hamiltonian formulations can be proved by direct calculations.

2. The model, its Lagrangian and Hamiltonian dynamics

In this paper we shall follow to the ADM parametrization for space-time metric:

$$ds^2 = (-N^2 + N_i N^i) dt^2 + 2N_i dt dx^i + g_{ij} dx^i dx^j. \quad (2.1)$$

Under the condition of spherical symmetry the metric is reduced to

$$\begin{aligned} ds^2 = & [-N^2(t, r) + (N^r(t, r))^2 V^2(t, r)] dt^2 + 2N^r(t, r) V^2(t, r) dt dr \\ & + V^2(t, r) dr^2 + W^2(t, r) (d\theta^2 + \sin^2 \theta d\varphi^2). \end{aligned} \quad (2.2)$$

where $N^r = N^1$ is the only component of the shift vector. In this model we have two gauge variables N and N^r which are fixed by two gauge conditions

$$N = f(V, W); \quad (2.3)$$

$$N^r = f^r(V, W). \quad (2.4)$$

$f(V, W)$, $f^r(V, W)$ are arbitrary functions. In differential form the gauge conditions will introduce missing velocities into the effective Lagrangian, so ensuring *an actual extension of phase space*:

$$\dot{N} = \frac{\partial f}{\partial V} \dot{V} + \frac{\partial f}{\partial W} \dot{W}; \quad (2.5)$$

$$\dot{N}^r = \frac{\partial f^r}{\partial V} \dot{V} + \frac{\partial f^r}{\partial W} \dot{W}. \quad (2.6)$$

We shall consider the Faddeev – Popov effective action including gauge and ghost sectors as it appears in the path integral approach to gauge field theories,

$$S_{(eff)} = S_{(grav)} + S_{(gauge)} + S_{(ghost)}. \quad (2.7)$$

The gravitational part of the effective action

$$S_{(grav)} = \int d^4x \sqrt{-g} R \quad (2.8)$$

is invariant under gauge transformations (1.1), but the gravitational Lagrangian involves second derivatives of metric components. To get field equations it is much easier to make use of the Lagrangian which is quadratic in first derivatives of metric components and can be obtained from the original one by omitting total derivatives. However, we shall have to return to the original gravitational Lagrangian when deriving the BRST charge in accordance with the Noether theorem (see Section 4).

The gauge-fixing part of the action is

$$S_{(gauge)} = \int dt \int_0^\infty dr \left[\lambda_0 \left(\dot{N} - \frac{\partial f}{\partial V} \dot{V} - \frac{\partial f}{\partial W} \dot{W} \right) + \lambda_r \left(\dot{N}^r - \frac{\partial f^r}{\partial V} \dot{V} - \frac{\partial f^r}{\partial W} \dot{W} \right) \right]. \quad (2.9)$$

Taking into account gauge transformations for gravitational variables

$$\delta N = -\dot{N}\eta^0 - N'\eta^r - N\dot{\eta}^0 + NN^r(\eta^0)'; \quad (2.10)$$

$$\delta N^r = -\dot{N}^r\eta^0 - (N^r)'\eta^r - N^r\dot{\eta}^0 - \dot{\eta}^r + N^r(\eta^r)' + \frac{N^2}{V^2}(\eta^0)' + (N^r)^2(\eta^0)'; \quad (2.11)$$

$$\delta V = -\dot{V}\eta^0 - V'\eta^r - V(\eta^r)' - VN^r(\eta^0)'; \quad (2.12)$$

$$\delta W = -\dot{W}\eta^0 - W'\eta^r, \quad (2.13)$$

that follow from (1.1), we get the Faddeev – Popov ghost action in the form

$$\begin{aligned} S_{(ghost)} = & \int dt \int_0^\infty dr \left[\bar{\theta}_0 \frac{d}{dt} \left(-\dot{N}\theta^0 - N'\theta^r - N\dot{\theta}^0 + NN^r(\theta^0)' \right. \right. \\ & - \frac{\partial f}{\partial V} \left[-\dot{V}\theta^0 - V'\theta^r - V(\theta^r)' - VN^r(\theta^0)' \right] - \frac{\partial f}{\partial W} \left[-\dot{W}\theta^0 - W'\theta^r \right] \\ & + \bar{\theta}_r \frac{d}{dt} \left(-\dot{N}^r\theta^0 - (N^r)'\theta^r - N^r\dot{\theta}^0 - \dot{\theta}^r + N^r(\theta^r)' + \frac{N^2}{V^2}(\theta^0)' + (N^r)^2(\theta^0)' \right. \\ & \left. \left. - \frac{\partial f^r}{\partial V} \left[-\dot{V}\theta^0 - V'\theta^r - V(\theta^r)' - VN^r(\theta^0)' \right] - \frac{\partial f^r}{\partial W} \left[-\dot{W}\theta^0 - W'\theta^r \right] \right) \right] \end{aligned} \quad (2.14)$$

$\bar{\theta}_0, \theta^0, \bar{\theta}_r, \theta^r$ are ghost variables. After redefinition

$$\pi_N = \lambda_0 + \dot{\bar{\theta}}_0\theta^0; \quad \pi_{N^r} = \lambda_r + \dot{\bar{\theta}}_r\theta^0 \quad (2.15)$$

we can write the effective Lagrangian in the form without second derivatives:

$$\begin{aligned} S_{(eff)} = & \int dt \int_0^\infty dr \left(\frac{\dot{V}\dot{W}W}{N} + \frac{V\dot{W}^2}{2N} - \frac{N'W'W}{V} - \frac{N(W')^2}{2V} - \frac{NV}{2} \right. \\ & - \frac{W'\dot{W}VN^r}{N} - \frac{WW'\dot{V}N^r}{N} - \frac{W\dot{W}V'N^r}{N} - \frac{W\dot{W}V(N^r)'}{N} \\ & \left. + \frac{WW'V'(N^r)^2}{N} + \frac{WW'VN^r(N^r)'}{N} + \frac{(W')^2V(N^r)^2}{2N} \right) \end{aligned}$$

$$\begin{aligned}
& + \pi_N \left(\dot{N} - \frac{\partial f}{\partial V} \dot{V} - \frac{\partial f}{\partial W} \dot{W} \right) + \pi_{N^r} \left(\dot{N}^r - \frac{\partial f^r}{\partial V} \dot{V} - \frac{\partial f^r}{\partial W} \dot{W} \right) \\
& + \dot{\theta}_0 \theta^r \left(N' - \frac{\partial f}{\partial V} V' - \frac{\partial f}{\partial W} W' \right) \\
& + \dot{\theta}_0 \left(N \dot{\theta}^0 - N N^r (\theta^0)' - \frac{\partial f}{\partial V} V N^r (\theta^0)' - \frac{\partial f}{\partial V} V (\theta^r)' \right) \\
& + \dot{\theta}_r \left[N^r \dot{\theta}^0 - \left(\frac{N^2}{V^2} + (N^r)^2 \right) (\theta^0)' + \dot{\theta}^r - N^r (\theta^r)' + (N^r)' \theta^r \right. \\
& \left. - \frac{\partial f^r}{\partial V} (V N^r (\theta^0)' + V (\theta^r)' + V' \theta^r) - \frac{\partial f^r}{\partial W} W' \theta^r \right] \Big) \quad (2.16)
\end{aligned}$$

Variation of the effective action with respect to N , N^r , V , W yields the Einstein equations for the model with additional terms resulting from the gauge-fixing and ghost parts of the action:

$$R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R = \kappa \left(T_{\mu(mat)}^\nu + T_{\mu(obs)}^\nu + T_{\mu(ghost)}^\nu \right), \quad (2.17)$$

where $T_{\mu(mat)}^\nu$ stands for the energy-momentum tensor of matter fields which can be included into the model, $T_{\mu(obs)}^\nu$ and $T_{\mu(ghost)}^\nu$ are obtained by varying the gauge-fixing and ghost action, respectively, so that $T_{\mu(obs)}^\nu$ describes the observer in a reference frame. It is clear that $T_{\mu(obs)}^\nu$ and $T_{\mu(ghost)}^\nu$ are not true tensors since they depend on chosen gauge conditions. Equations (2.17) can be called *the gauged Einstein equations*. By adding ghost equations and gauge conditions (2.5), (2.6) to the gauged Einstein equations, we obtain *the extended set of Lagrangian equations* for our model which is presented in Appendix A. The explicit expressions of $T_{\mu(obs)}^\nu$, $T_{\mu(ghost)}^\nu$ are given in Appendix C. Below $\kappa = 8\pi$ (in special units when $G = c = \hbar = 1$).

Now we can find the momenta conjugate to all gravitational and ghost variables:

$$P_N = \pi_N; \quad (2.18)$$

$$P_{N^r} = \pi_{N^r}; \quad (2.19)$$

$$P_V = \frac{W \dot{W}}{N} - \frac{W' W N^r}{N} - \pi_N \frac{\partial f}{\partial V} - \pi_{N^r} \frac{\partial f^r}{\partial V}; \quad (2.20)$$

$$P_W = \frac{W \dot{V}}{N} + \frac{V \dot{W}}{N} - \frac{W' V N^r}{N} - \frac{W V' N^r}{N} - \frac{V W (N^r)'}{N} - \pi_N \frac{\partial f}{\partial W} - \pi_{N^r} \frac{\partial f^r}{\partial W}; \quad (2.21)$$

$$P_{\theta_0} = N' \theta^r - \frac{\partial f}{\partial V} V' \theta^r - \frac{\partial f}{\partial W} W' \theta^r + N \dot{\theta}^0 - N N^r (\theta^0)' - \frac{\partial f}{\partial V} V N^r (\theta^0)' - \frac{\partial f}{\partial V} V (\theta^r)'; \quad (2.22)$$

$$\bar{P}_{\theta_0} = \dot{\theta}_0 N + \dot{\theta}_r N^r; \quad (2.23)$$

$$\begin{aligned}
P_{\theta_r} &= N^r \dot{\theta}^0 - \frac{N^2}{V^2} (\theta^0)' - (N^r)^2 (\theta^0)' + \dot{\theta}^r - N^r (\theta^r)' + (N^r)' \theta^r \\
&- \frac{\partial f^r}{\partial V} V N^r (\theta^0)' - \frac{\partial f^r}{\partial V} V (\theta^r)' - \frac{\partial f^r}{\partial V} V' \theta^r - \frac{\partial f^r}{\partial W} W' \theta^r; \quad (2.24)
\end{aligned}$$

$$\bar{P}_{\theta_r} = \dot{\theta}_r. \quad (2.25)$$

Introducing of the missing velocities by means of the differential form of gauge conditions (2.5), (2.6) enables us to construct a Hamiltonian in extended phase space not applying to the Dirac procedure, by the usual rule

$$H = \int_0^\infty dr \left(P_N \dot{N} + P_{N^r} \dot{N}^r + P_V \dot{V} + P_W \dot{W} + \bar{P}_{\theta_0} \dot{\theta}^0 + \dot{\theta}_0 P_{\theta_0} + \bar{P}_{\theta_r} \dot{\theta}^r + \dot{\theta}_r P_{\theta_r} - L \right); \quad (2.26)$$

$$\begin{aligned}
H = & \int_0^\infty dr \left[\frac{N}{W} P_V P_W - \frac{NV}{2W^2} P_V^2 + P_V V' N^r + P_V V (N^r)' + P_W W' N^r + \frac{N' W' W}{V} + \frac{N(W')^2}{2V} + \frac{NV}{2} \right. \\
& + P_N \frac{\partial f}{\partial V} V' N^r + P_N \frac{\partial f}{\partial W} W' N^r + P_N \frac{\partial f}{\partial V} V (N^r)' + P_{N^r} \frac{\partial f^r}{\partial V} V' N^r + P_{N^r} \frac{\partial f^r}{\partial W} W' N^r \\
& + P_{N^r} \frac{\partial f^r}{\partial V} V (N^r)' + \frac{N}{W} P_V P_N \frac{\partial f}{\partial W} + \frac{N}{W} P_W P_N \frac{\partial f}{\partial V} - \frac{NV}{W^2} P_V P_N \frac{\partial f}{\partial V} \\
& + \frac{N}{W} P_V P_{N^r} \frac{\partial f^r}{\partial W} + \frac{N}{W} P_W P_{N^r} \frac{\partial f^r}{\partial V} - \frac{NV}{W^2} P_V P_{N^r} \frac{\partial f^r}{\partial V} - \frac{NV}{2W^2} P_N^2 \left(\frac{\partial f}{\partial V} \right)^2 \\
& - \frac{NV}{2W^2} P_{N^r}^2 \left(\frac{\partial f^r}{\partial V} \right)^2 + \frac{N}{W} P_N^2 \frac{\partial f}{\partial V} \frac{\partial f}{\partial W} + \frac{N}{W} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial f^r}{\partial W} \\
& + \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial W} + \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial W} \frac{\partial f^r}{\partial V} - \frac{NV}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V} \\
& + \frac{1}{N} \bar{P}_{\theta^0} P_{\bar{\theta}_0} + \bar{P}_{\theta^r} P_{\bar{\theta}_r} - \frac{N^r}{N} \bar{P}_{\theta^r} P_{\bar{\theta}_0} - \frac{N'}{N} \bar{P}_{\theta^0} \theta^r + N^r \bar{P}_{\theta^0} (\theta^0)' + N^r \bar{P}_{\theta^r} (\theta^r)' \\
& - (N^r)' \bar{P}_{\theta^r} \theta^r + \frac{N' N^r}{N} \bar{P}_{\theta^r} \theta^r + \frac{N^2}{V^2} \bar{P}_{\theta^r} (\theta^0)' + \frac{\partial f^r}{\partial V} V' \bar{P}_{\theta^r} \theta^r + \frac{\partial f^r}{\partial W} W' \bar{P}_{\theta^r} \theta^r \\
& + \frac{\partial f^r}{\partial V} V N^r \bar{P}_{\theta^r} (\theta^0)' + \frac{\partial f^r}{\partial V} V \bar{P}_{\theta^r} (\theta^r)' + \frac{V}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^r)' + \frac{V N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^0)' \\
& - \frac{V N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^r)' - \frac{V (N^r)^2}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^0)' + \frac{V'}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} \theta^r \\
& \left. - \frac{V' N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} \theta^r + \frac{W'}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^0} \theta^r - \frac{W' N^r}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^r} \theta^r \right]. \quad (2.27)
\end{aligned}$$

The first line in (2.27) is the Hamiltonian for pure gravity that can be presented as a linear combination of Dirac secondary constraints since it is believed that a full derivative with respect to r can be omitted in this expression:

$$H_D = \int_0^\infty dr \left[N \left(\frac{1}{W} P_V P_W - \frac{V}{2W^2} P_V^2 - \frac{WW''}{V} - \frac{(W')^2}{2V} + \frac{V'WW'}{V^2} + \frac{V}{2} \right) + N^r (P_W W' - P_V' V) \right]. \quad (2.28)$$

However, as it follows from (2.27), the Hamiltonian in extended phase space cannot be written down as a linear combination of constraints. Now we can write down the set of Hamiltonian equations in extended phase space presented explicitly in Appendix B.

It is important to emphasize that in this formulation of Hamiltonian dynamics the constraints as well as the gauge conditions have the status of Hamiltonian equations. Indeed, the Hamiltonian equations (B.1), (B.3) coincide with the gauge conditions (A.9), (A.10), while the equations (B.2), (B.4) reproduce the constraints (A.1), (A.2) in the Lagrangian formalism. The equations (B.5) – (B.8) for physical gravitational degrees of freedom after some rearrangement can be shown to be equivalent to the dynamical Lagrangian equations (A.3), (A.4), and Eqs. (B.9) – (B.16) are equivalent to the ghost equations (A.5) – (A.8). Thus, in this Section we have got two sets of extended equations for our spherically symmetric model in the Lagrangian and Hamiltonian formalisms, which are proved to be completely equivalent. In the next Section we shall consider some particular solutions to these equations.

3. Particular solutions to the extended set of equations

3.1. Solutions to gauge invariant Einstein equations. It is obvious enough that if we put all ghost as well as the Lagrange multipliers π_N , π_{N^r} equal to zero in the extended set of equations (A.1) – (A.8), we would return to gauge invariant Einstein equations for the model. The ghost equations (A.5) – (A.8) are satisfied identically, so in this case we have just four equations (A.1) – (A.4) and two gauge conditions (A.9), (A.10). For further simplification of the equations we choose the condition (2.4) in the form

$$N^r = 0. \quad (3.1)$$

Taking into account (3.1) we come to the following set of equations:

$$0 = \frac{\dot{V}W\dot{W}}{N^2} + \frac{V\dot{W}^2}{2N^2} + \frac{V}{2} - \frac{(W')^2}{2V} - \frac{WW''}{V} + \frac{V'W'W}{V^2}; \quad (3.2)$$

$$0 = \frac{WW'\dot{V}}{N} - \frac{WV\dot{W}'}{N} + \frac{N'WV\dot{W}}{N^2}; \quad (3.3)$$

$$0 = \frac{\dot{W}^2}{2N} + \frac{W\ddot{W}}{N} - \frac{W\dot{W}\dot{N}}{N^2} + \frac{N}{2} - \frac{N(W')^2}{2V^2} - \frac{N'W'W}{V^2}; \quad (3.4)$$

$$0 = \frac{W\ddot{V}}{N} + \frac{\dot{V}\dot{W}}{N} + \frac{V\ddot{W}}{N} - \frac{\dot{N}\dot{V}W}{N^2} - \frac{\dot{N}\dot{W}V}{N^2} - \frac{WN''}{V} + \frac{WN'V'}{V^2} - \frac{W'N'}{V} + \frac{W'NV'}{V^2} - \frac{W''N}{V}; \quad (3.5)$$

$$0 = N - f(V, W). \quad (3.6)$$

Solutions to Eqs. (3.2) – (3.5) can be considered as solutions of gauge invariant Einstein equations for the spherically symmetric model, though any solution requires fixing of gauge conditions to be presented in its final form. Eqs. (3.2), (3.4), (3.5) can be obtained from a Lagrangian for the model with the metric (2.2) where $N^r(t, r) = 0$. Let us note that Eq. (3.3) which is equivalent to the $\binom{0}{1}$ Einstein equation could not be obtained if one just substitutes (3.1) to the gravitational part of the action. The second gauge condition (3.6) can be imposed on different stages of solving the equations, but in what following we shall start from fixing this condition.

The Hamiltonian set of equations will be also rather simplified:

$$\dot{N} = \frac{N}{W}P_V\frac{\partial f}{\partial W} + \frac{N}{W}P_W\frac{\partial f}{\partial V} - \frac{NV}{W^2}P_V\frac{\partial f}{\partial V}; \quad (3.7)$$

$$0 = -\frac{1}{W}P_VP_W + \frac{V}{2W^2}P_V^2 + \frac{W''W}{V} - \frac{V'W'W}{V^2} + \frac{(W')^2}{2V} - \frac{V}{2}; \quad (3.8)$$

$$\dot{N}^r = 0; \quad (3.9)$$

$$0 = (P_V)'V - P_WW'; \quad (3.10)$$

$$\dot{V} = \frac{N}{W}P_W - \frac{NV}{W^2}P_V; \quad (3.11)$$

$$\dot{P}_V = \frac{N}{2W^2}P_V^2 + \frac{N'W'W}{V^2} + \frac{N(W')^2}{2V^2} - \frac{N}{2}; \quad (3.12)$$

$$\dot{W} = \frac{N}{W}P_V; \quad (3.13)$$

$$\dot{P}_W = \frac{N}{W^2} P_V P_W - \frac{NV}{W^3} P_V^2 + \frac{N''W}{V} - \frac{N'V'W}{V^2} + \frac{N'W'}{V} + \frac{NW''}{V} - \frac{NV'W'}{V^2}. \quad (3.14)$$

3.1.1. The Schwarzschild solution. The Schwarzschild solution [17] is probably the most famous spherically symmetric solution to the Einstein equations. First of all, we choose the gauge condition (3.6) in the form

$$N = \frac{1}{V}. \quad (3.15)$$

Following to Schwarzschild [17], we shall seek for a solution that does not depend on time. Then, Eq. (3.3) is identically satisfied, and Eqs. (3.2), (3.4) under the condition (3.15) look like

$$0 = \frac{V}{2} - \frac{(W')^2}{2V} - \frac{WW''}{V} + \frac{V'W'W}{V^2}; \quad (3.16)$$

$$0 = \frac{V}{2} - \frac{(W')^2}{2V} + \frac{V'W'W}{V^2}. \quad (3.17)$$

From (3.16), (3.17) we get

$$W'' = 0; \quad W(r) = C_1 r + C_2. \quad (3.18)$$

One can put $C_1 = 1$, $C_2 = 0$, so that the length of a circle with a center in coordinate origin would be $2\pi r$. Then $W(r) = r$. Eq. (3.17) takes the form

$$0 = \frac{V}{2} - \frac{1}{2V} + \frac{V'r}{V^2}. \quad (3.19)$$

Its solution can be written as

$$V(r) = \frac{1}{\sqrt{1 - \frac{2GM}{r}}}, \quad (3.20)$$

where M is associated with a Schwarzschild mass. Eq. (3.5) is identically satisfied. It is easy to see that the same solution follows from Eqs. (3.7) – (3.14). In our consideration we have started from fixing the gauge condition which helps to separate out the required solution, meanwhile usually a gauge is imposed on a final stage of the procedure. So we get the Schwarzschild metric:

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3.21)$$

3.1.2. The Lemaitre - Tolman – Bondi metric. The Lemaitre - Tolman – Bondi solution is the solution for dust matter in synchronous and comoving reference frame (see, for example, [18] and references therein). Actually one should add energy-momentum tensor of dust into the equations. In this case the only non-zero component of the matter energy-momentum tensor is $T_{0(mat)}^0 = \varepsilon$, where ε is the energy density of dust. After imposing the gauge condition

$$N = 1 \quad (3.22)$$

Eq. (3.2) can be rewritten as

$$\frac{2\dot{V}\dot{W}}{VW} + \frac{\dot{W}^2}{W^2} + \frac{1}{W^2} - \frac{(W')^2}{V^2 W^2} - \frac{2W''}{V^2 W} + \frac{2V'W'}{V^3 W} = 8\pi\varepsilon. \quad (3.23)$$

Eq. (3.3) gives

$$0 = W'\dot{V} - V\dot{W}'. \quad (3.24)$$

Its solution yields a relation between $V(t, r)$ and $W(t, r)$,

$$V(t, r) = C_1(r)W'(t, r), \quad (3.25)$$

where $C_1(r)$ is an arbitrary function. The solution (3.25) can be obtained from Hamiltonian equations. From (3.13), (3.11) one can express

$$P_V = W\dot{W}, \quad P_W = W\dot{V} + V\dot{W}. \quad (3.26)$$

Substitution (3.26) into (3.10) gives (3.25). Finally, the Lemaitre - Tolman - Bondi metric can be presented as

$$ds^2 = -dt^2 + (C_1(r)W'(t, r))^2 dr^2 + W^2(t, r) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (3.27)$$

Eq. (3.4) after substitution (3.25) leads to

$$\dot{W}^2 = \frac{C_2(r)}{W} + \frac{1}{C_1^2(r)} - 1, \quad (3.28)$$

$C_2(r)$ is another arbitrary function. The substitution of (3.25) into (3.23) together with (3.28) gives the solution for dust energy density

$$\varepsilon = \frac{1}{8\pi} \frac{C_2'(r)}{W'W^2}. \quad (3.29)$$

3.1.3. The Friedmann closed isotropic model. We can obtain the Einstein equations for the Friedmann closed isotropic model as a particular case of Eqs. (3.2) – (3.6). We then should put

$$V(t, r) = a(t), \quad W(t, r) = a(t) \sin r. \quad (3.30)$$

Eq. (3.2) gives the constraint

$$\frac{a\dot{a}^2}{N^2} + a = 0. \quad (3.31)$$

Eq. (3.3) is identically satisfied. Eqs. (3.4), (3.5) are both reduced to the dynamical equation

$$\frac{a\ddot{a}}{N} + \frac{\dot{a}^2}{2N} - \frac{a\dot{a}\dot{N}}{N^2} + \frac{N}{2} = 0. \quad (3.32)$$

Eqs. (3.31), (3.32) should be complemented by a gauge condition $N = f(a)$, or

$$\dot{N} = \frac{df}{da} \dot{a}. \quad (3.33)$$

The same equations can be obtained from the Hamiltonian set (3.7) – (3.14). From (3.13), (3.11) we get

$$P_V = \frac{W\dot{W}}{N} = \frac{a\dot{a}}{N} \sin^2 r, \quad P_W = \frac{W\dot{V}}{N} + \frac{V\dot{W}}{N} = \frac{2a\dot{a}}{N} \sin r. \quad (3.34)$$

Eq. (3.7) can be rewritten in the form (3.33) keeping in mind that

$$\frac{df}{da} = \frac{\partial f}{\partial V} + \frac{\partial f}{\partial W} \sin r. \quad (3.35)$$

Eq. (3.8) leads to the constraint (3.31). Eq. (3.10) is identically satisfied. Eqs. (3.12), (3.14) can be both reduced to the equation (3.32).

3.2. A possible role of gauge-noninvariant terms. In this section we shall discuss the role of Lagrange multipliers and ghosts in the extended set of Lagrangian equations, and consider a restricted class of solutions when $\theta^0, \bar{\theta}_0$ are functions of t only while $\theta^r, \bar{\theta}_r$ are functions of r . We shall show that even for this restricted class their role can be very different: For example, to obtain the Schwarzschild metric one should choose trivial solution for Lagrange multipliers and ghosts, otherwise the metric components will not have the Galilean form at infinity. On the contrary, in the case of the Lemaitre - Tolman - Bondi model gauge-noninvariant sector of the model (gauge-fixing and ghost components) can imitate dust matter with zero pressure and energy density given by (3.29). In the end of this section we shall refer to our earlier papers where the role of gauge-noninvariant sector has been analyzed for systems with finite number degrees of freedom.

As it was done in Section 3.1, for simplicity we choose one of the gauge conditions in the form $N^r = 0$ (3.1). Under this condition Eq. (A.7) reads

$$\ddot{\theta}^r - \frac{d}{dt} \left(\frac{N^2}{V^2} (\theta^0)' \right) = 0. \quad (3.36)$$

This equation is satisfied if

$$\theta^0 = \theta^0(t), \quad \theta^r = \theta^r(r). \quad (3.37)$$

Indeed, the condition (3.1) imposes limitations on possible coordinates transformations, so that the admissible transformations of t, r are $\tau = \tau(t), \rho = \rho(r)$. The infinitesimal parameters η^0, η^r in (2.10) – (2.13) must be functions of t, r , correspondingly: $\eta^0 = \eta^0(t), \eta^r = \eta^r(r)$. Since ghost variables are known to compensate residual gauge transformations, it is reasonable to consider ghost solutions of the form (3.37).

Keeping in mind (3.1) and (3.37), one can rewrite Eq. (A.5) as

$$\frac{d}{dt} \left[N \dot{\theta}^0 - \frac{\partial f}{\partial V} V (\theta^r)' + \left(N' - \frac{\partial f}{\partial V} V' - \frac{\partial f}{\partial W} W' \right) \theta^r \right] = 0. \quad (3.38)$$

As we remember, Eq. (A.9) is the differential form of Eq. (2.3). As a consequence of the latter we get

$$N' = \frac{\partial f}{\partial V} V' + \frac{\partial f}{\partial W} W', \quad (3.39)$$

and Eq. (3.38) looks like

$$\frac{d}{dt} \left[N \dot{\theta}^0 - \frac{\partial f}{\partial V} V (\theta^r)' \right] = 0. \quad (3.40)$$

Under the condition (3.1) equations for ghosts $\bar{\theta}_0, \bar{\theta}_r$ (A.6), (A.8) will be reduced to

$$\frac{d}{dt} \left(\dot{\bar{\theta}}_0 N \right) - \frac{d}{dr} \left(\dot{\bar{\theta}}_r \frac{N^2}{V^2} \right) = 0 \quad (3.41)$$

and

$$\frac{d}{dr} \left(\dot{\bar{\theta}}_0 \frac{\partial f}{\partial V} V \right) - \ddot{\bar{\theta}}_r = 0. \quad (3.42)$$

Again, let us consider the solutions such that

$$\bar{\theta}^0 = \bar{\theta}^0(t), \quad \bar{\theta}^r = \bar{\theta}^r(r). \quad (3.43)$$

Thus, the two equations for $\bar{\theta}_0, \bar{\theta}_r$ are

$$\frac{d}{dt}(\dot{\bar{\theta}}_0 N) = 0, \quad \frac{d}{dr}\left(\dot{\bar{\theta}}_0 \frac{\partial f}{\partial V} V\right) = 0. \quad (3.44)$$

One can notice that π_{Nr} enters into Eqs. (A.3), (A.4) being multiplied by the derivatives of the function f^r which is equal to zero; π_{Nr} also enters into Eq. (A.2). The ghost terms are missed out from last equation if (3.37), (3.43) are true. One can put

$$\pi_{Nr} = 0. \quad (3.45)$$

In this case Eq. (A.2) will be reduced to the constraint (3.3).

Applying the Bianchi identity to the both sides of (2.17), we obtain

$$\left(T_{\mu(mat)}^\nu + T_{\mu(obs)}^\nu + T_{\mu(ghost)}^\nu\right)_{;\nu} = 0. \quad (3.46)$$

We can require that the conditions $T_{\mu;\nu}^\nu = 0$ would be satisfied separately by the energy-momentum tensor of matter fields and quasi-tensors of the observer and of ghosts. It will give us equations for the Lagrange multiplier π_N . Taking into account that $N^r = 0$, $\pi_{Nr} = 0$, the non-zero components of $T_{\mu(obs)}^\nu$ are

$$T_{0(obs)}^0 = \frac{1}{4\pi} \dot{\pi}_N \frac{1}{VW^2}; \quad (3.47)$$

$$T_{1(obs)}^1 = -\frac{1}{4\pi} \dot{\pi}_N \frac{1}{NW^2} \frac{\partial f}{\partial V}; \quad (3.48)$$

$$T_{2(obs)}^2 = T_{3(obs)}^3 = -\frac{1}{8\pi} \dot{\pi}_N \frac{1}{NVW} \frac{\partial f}{\partial W}. \quad (3.49)$$

Equations $T_{\mu(obs);\nu}^\nu = 0$ with $\mu = 0, 1$ give

$$\frac{d}{dt}(\dot{\pi}_N N) = 0, \quad \frac{d}{dr}\left(\dot{\pi}_N \frac{\partial f}{\partial V} V\right) = 0. \quad (3.50)$$

The non-zero components of $T_{\mu(ghost)}^\nu$ are

$$T_{0(ghost)}^0 = \frac{1}{4\pi} \frac{1}{VW^2} \left[\dot{\bar{\theta}}_0(\theta^r)' - \dot{\bar{\theta}}_0 \dot{\theta}^0\right]; \quad (3.51)$$

$$T_{1(ghost)}^1 = \frac{1}{4\pi} \frac{V}{NW^2} \frac{\partial^2 f}{\partial V^2} \dot{\bar{\theta}}_0(\theta^r)'; \quad (3.52)$$

$$T_{2(ghost)}^2 = T_{3(ghost)}^3 = \frac{1}{8\pi} \left[\frac{1}{NW} \frac{\partial^2 f}{\partial V \partial W} \dot{\bar{\theta}}_0(\theta^r)' - \frac{1}{NVW} \frac{\partial f}{\partial W} \dot{\bar{\theta}}_0(\theta^r)'\right]. \quad (3.53)$$

As one can check, the equations $T_{\mu(ghost);\nu}^\nu = 0$ are completely compatible with Lagrangian equations for ghosts.

3.2.1. The Schwarzschild metric. In this case the consideration of equations for ghost and the Lagrange multiplier π_N does not lead to non-zero solutions for this variables, at least under requirements (3.37), (3.43), (3.45). Indeed, Eqs. (3.44) result in $\dot{\bar{\theta}}_0 = 0$, and all components of $T_{\mu(ghost)}^\nu$ are zero, thus the ghost sector gives no contribution. It follows from Eqs. (3.50) that

$$\frac{\dot{\pi}_N}{V} = \text{const} = \frac{C_3}{2}. \quad (3.54)$$

and non-zero components of $T_{\mu(obs)}^\nu$ are

$$T_{0(obs)}^0 = T_{1(obs)}^1 = \frac{1}{8\pi} \frac{C_3}{W^2}. \quad (3.55)$$

Then instead of (3.20) we get

$$V(r) = \frac{1}{\sqrt{1 + C_3 - \frac{2GM}{r}}}. \quad (3.56)$$

The requirement for metric components to have the Galilean form at infinity means that $C_3 = 0$ and all components of $T_{\mu(obs)}^\nu$ are zero.

3.2.2. The Lemaitre - Tolman – Bondi solution. This solution is rather interesting. From (3.40), (3.44) and (3.50) we obtain

$$\dot{\bar{\theta}}_0 = \text{const}; \quad \dot{\theta}_0 = \text{const}; \quad \dot{\pi}_N = \varphi(r), \quad (3.57)$$

and, after the substitution of (3.25), for non-zero components of $T_{\mu(obs)}^\nu, T_{\mu(ghost)}^\nu$ we have

$$T_{0(obs)}^0 = \frac{1}{4\pi} \frac{\varphi_1(r)}{W'W^2}; \quad T_{0(ghost)}^0 = \frac{1}{4\pi} \frac{\varphi_2(r)}{W'W^2}, \quad (3.58)$$

$\varphi(r), \varphi_1(r), \varphi_2(r)$ are arbitrary functions. We can see that the both quasi-tensors $T_{\mu(ghost)}^\nu, T_{\mu(obs)}^\nu$ describe a medium with zero pressure and energy density of the form (3.29). The meaningful feature of this consideration is that we do not need to introduce an extra dust matter into the model, while a dust matter is simulated by its ghost and gauge-fixing components.

3.2.3. Models with finite numbers of degrees of freedom. In our early papers ([16] and others) it was demonstrated for models with finite numbers of degrees of freedom that the quasi-tensor $T_{\mu(obs)}^\nu$ describes a continual medium with the equation of state essentially depending on the parametrization of the gauge variable and the gauge condition for it. It is easy to see that a change of the gauge variable (the choice of its parametrization), which looks like $N = v(\tilde{N}, a)$ for the isotropic model, together with an additional condition for a new variable \tilde{N} , say, $\tilde{N} = 1$, fixes the gauge condition for N : $f(a) = v(1, a)$. It has been shown in [19] that the choice of parametrization function $v(\tilde{N}, a) = \tilde{N}a^{n-3}$ (which is equivalent to the gauge condition $N = a^{n-3}$) results in the equation of state $p = \left(\frac{n}{3} - 1\right)\varepsilon$, where n can have various values, for example, $n = 0$ corresponds to de Sitter false vacuum, $n = 4$ corresponds a radiation dominated universe, etc. It confirms that at least in some cases one can simulate a medium with required properties only by means of introducing a suitable parametrization and gauge conditions. Another component of the integrated system, which includes pure gravity, matter fields and the observer, is ghost fields; analyzing its quasi-tensor $T_{\mu(ghost)}^\nu$ for a given gauge one can obtain its equation of state. However, the role of ghost fields has not been studied enough.

4. The BRST charge

It is known that the Faddeev – Popov effective action possesses a residual global symmetry, the so-called BRST symmetry. In the Lagrangian formalism the BRST transformations for our model are given by (2.10) – (2.13),

where infinitesimal parameters η^μ should be replaced by $\bar{\varepsilon}\theta^\mu$, $\bar{\varepsilon}$ is a Grassmannian parameter,

$$\delta N = \bar{\varepsilon} \left[-\dot{N}\theta^0 - N'\theta^r - N\dot{\theta}^0 + NN^r(\theta^0)' \right]; \quad (4.1)$$

$$\delta N^r = \bar{\varepsilon} \left[-\dot{N}^r\theta^0 - (N^r)'\theta^r - N^r\dot{\theta}^0 - \dot{\theta}^r + N^r(\theta^r)' + \frac{N^2}{V^2}(\theta^0)' + (N^r)^2(\theta^0)' \right]; \quad (4.2)$$

$$\delta V = \bar{\varepsilon} \left[-\dot{V}\theta^0 - V'\theta^r - V(\theta^r)' - VN^r(\theta^0)' \right]; \quad (4.3)$$

$$\delta W = \bar{\varepsilon} \left[-\dot{W}\theta^0 - W'\theta^r \right]. \quad (4.4)$$

Moreover,

$$\delta\theta^0 = \bar{\varepsilon} \left[\dot{\theta}^0\theta^0 + (\theta^0)'\theta^r \right]; \quad (4.5)$$

$$\delta\theta^r = \bar{\varepsilon} \left[\dot{\theta}^r\theta^0 + (\theta^r)'\theta^r \right]; \quad (4.6)$$

$$\delta\bar{\theta}_0 = -\bar{\varepsilon}\lambda_0; \quad (4.7)$$

$$\delta\bar{\theta}_r = -\bar{\varepsilon}\lambda_r; \quad (4.8)$$

$$\delta\lambda_0 = 0; \quad (4.9)$$

$$\delta\lambda_r = 0. \quad (4.10)$$

The transformations (4.1) – (4.10) should be supplemented by coordinated transformations

$$\delta t = \bar{\varepsilon}\theta^0; \quad \delta r = \bar{\varepsilon}\theta^r. \quad (4.11)$$

As a consequence of a global symmetry there exists a BRST charge which plays a role of a generator of BRST transformations in extended phase space. As we have already mentioned, in the BFV approach it is constructed as a series in powers of Grassmannian variables with coefficients given by generalized structure functions of constraints algebra:

$$\Omega = c^\alpha U_\alpha^{(0)} + c^\beta c^\gamma U_{\gamma\beta}^{(1)\alpha} \bar{\rho}_\alpha + \dots \quad (4.12)$$

c^α , $\bar{\rho}_\alpha$ are BFV ghosts, $U^{(n)}$ are n th order structure functions, while zero order structure functions $U_\alpha^{(0)}$ are Dirac secondary constraints.

Since the BFV prescription of constructing the BRST charge is essentially rely upon constraints algebra, it cannot produce correct transformations for gauge gravitational variables, like a linear combination of constraints cannot produce them in the Dirac approach. In [1] we have analyzed the algorithm suggested in [20] that aims at modifying the Dirac scheme and constructing a generator producing correct transformations for all variables. However, this algorithm fails to be applied to an arbitrary parametrization of gravitational variables, so it is not general enough and cannot be considered as a required solution to the problem.

At the same time, the existence of global BRST symmetry enables us to propose another method based upon the Noether theorem and the equivalence of Lagrangian dynamics and Hamilton dynamics in extended phase space. In this section we shall apply it to our spherically symmetric model, however nothing prevent one from applying it to any other gravitational model including the full theory of gravity. The fact that gauge

degrees of freedom are treated on the equal footing with other variables allows one to make transformations of variables including gauge ones which have been proved to be canonical in extended phase space [1] and do not affect the algebra of Poisson brackets. So, the proposed method will work for any reasonable parametrization of gravitational variables.

You can find the proof of BRST symmetry of the Faddeev – Popov effective action for Yang – Mills fields in any book on quantum field theory [21]. In the case of gravity we deal with space-time symmetry, and we should take into account explicit dependence of the Lagrangian and the measure on space-time coordinates. One can check that the sum of gauge-fixing and ghost parts of the action (2.9), (2.14) is not invariant under transformations (4.1) – (4.11). In some works the BRST invariance is guaranteed by asymptotical boundary conditions for ghosts and Lagrange multipliers [14, 22]. The legitimacy of asymptotic boundary conditions is questionable in the case of space-time of arbitrary topology. Therefore, we seek for a BRST invariant form of the action without appealing to any additional conditions. One can check that to ensure its BRST invariance we have to add to the action the following term containing only full derivatives and not affecting the set of equations obtained in Section 2:

$$S_{(add)} = \int dt \int_0^\infty dr \left(\frac{d}{dt} \left[\bar{\theta}_0 \left(\dot{N} - \frac{\partial f}{\partial V} \dot{V} - \frac{\partial f}{\partial W} \dot{W} \right) \theta^0 \right] + \frac{d}{dr} \left[\bar{\theta}_0 \left(\dot{N} - \frac{\partial f}{\partial V} \dot{V} - \frac{\partial f}{\partial W} \dot{W} \right) \theta^r \right] \right. \\ \left. + \frac{d}{dt} \left[\bar{\theta}_r \left(\dot{N}^r - \frac{\partial f^r}{\partial V} \dot{V} - \frac{\partial f^r}{\partial W} \dot{W} \right) \theta^0 \right] + \frac{d}{dr} \left[\bar{\theta}_r \left(\dot{N}^r - \frac{\partial f^r}{\partial V} \dot{V} - \frac{\partial f^r}{\partial W} \dot{W} \right) \theta^r \right] \right). \quad (4.13)$$

As was mentioned in Section 2, the gravitational part of the action in (2.16) is not invariant under gauge transformation and, therefore under BRST transformations (4.1) – (4.4). Then, we should return to the gravitational action (2.8). Now we deal with the Lagrangian which involves second derivatives of metric components and ghosts. The BRST charge is constructed in accordance with the Noether theorem generalized for theories with high derivatives:

$$\Omega = \int d^3x \left[\frac{\partial L}{\partial(\partial_0 \phi^a)} \delta \phi^a + \frac{\partial L}{\partial(\partial_0 \partial_\mu \phi^a)} \delta(\partial_\mu \phi^a) - \partial_\mu \left(\frac{\partial L}{\partial(\partial_0 \partial_\mu \phi^a)} \right) \delta \phi^a + \partial_0 (Lx^0) \right]. \quad (4.14)$$

ϕ^a stands for all variables N , N^r , V , W and ghosts. After some tedious calculations we come to the following expression for the BRST charge in the spherically symmetric model:

$$\Omega = \int dr \left[-\mathcal{H} \theta^0 - P_V V' \theta^r - P_N \frac{\partial f}{\partial V} V' \theta^r - P_{N^r} \frac{\partial f^r}{\partial V} V' \theta^r \right. \\ - P_W W' \theta^r - P_N \frac{\partial f}{\partial W} W' \theta^r - P_{N^r} \frac{\partial f^r}{\partial W} W' \theta^r \\ - P_V V N^r (\theta^0)' - P_N \frac{\partial f}{\partial V} V N^r (\theta^0)' - P_{N^r} \frac{\partial f^r}{\partial V} V N^r (\theta^0)' \\ - P_V V (\theta^r)' - P_N \frac{\partial f}{\partial V} V (\theta^r)' - P_{N^r} \frac{\partial f^r}{\partial V} V (\theta^r)' \\ \left. - \bar{P}_{\theta^0} (\theta^0)' \theta^r - \bar{P}_{\theta^r} (\theta^r)' \theta^r - P_N P_{\bar{\theta}_0} - P_{N^r} P_{\bar{\theta}_r} - \frac{N W W' (\theta^0)'}{V} \right], \quad (4.15)$$

\mathcal{H} is a Hamiltonian density in (2.27). It can be directly checked that the charge (4.15) generates transformations (4.1) – (4.8). Let us emphasized that the Hamiltonian equations in extended phase space, in particular,

constraints and gauge conditions which have the status of Hamiltonian equations, is used to get correct results, for instance,

$$\begin{aligned}
\delta N &= \{N, \bar{\varepsilon}\Omega\} = \bar{\varepsilon} \frac{\delta\Omega}{\delta P_N} \\
&= \bar{\varepsilon} \left[-\frac{\partial\mathcal{H}}{\partial P_N} \theta^0 - \frac{\partial f}{\partial V} V' \theta^r - \frac{\partial f}{\partial W} W' \theta^r - \frac{\partial f}{\partial V} V N^r (\theta^0)' - \frac{\partial f}{\partial V} V (\theta^r)' - P_{\bar{\theta}_0} \right] \\
&= \bar{\varepsilon} \left[-\dot{N} \theta^0 - N' \theta^r - N \dot{\theta}^0 + N N^r (\theta^0)' \right].
\end{aligned} \tag{4.16}$$

Here we used one of the Hamiltonian equations $\dot{N} = \frac{\delta H}{\delta P_N}$ (B.1), and the expression for $P_{\bar{\theta}_0}$ (2.22). To check (4.9), (4.10) one should firstly find $\delta P_N, \delta P_{N^r}$.

$$\begin{aligned}
\delta P_N &= \{P_N, \bar{\varepsilon}\Omega\} = -\bar{\varepsilon} \frac{\delta\Omega}{\delta N} = \bar{\varepsilon} \left[-\frac{\partial\Omega}{\partial N} + \left(\frac{\partial\Omega}{\partial N'} \right)' \right] \\
&= \bar{\varepsilon} \left[\frac{\partial\mathcal{H}}{\partial N} \theta^0 - \left(\frac{\partial\mathcal{H}}{\partial N'} \theta^0 \right)' + \frac{W W' (\theta^0)'}{V} \right] \\
&= \bar{\varepsilon} \left[\left(\frac{\partial\mathcal{H}}{\partial N} - \left(\frac{\partial\mathcal{H}}{\partial N'} \right)' \right) \theta^0 - \frac{\partial\mathcal{H}}{\partial N'} (\theta^0)' + \frac{W W' (\theta^0)'}{V} \right] \\
&= \bar{\varepsilon} \left[-\dot{P}_N \theta^0 - \left(\frac{W W'}{V} - \frac{1}{N} \bar{P}_{\theta^0} \theta^r + \frac{N^r}{N} \bar{P}_{\theta^r} \theta^r \right) (\theta^0)' + \frac{W W' (\theta^0)'}{V} \right] \\
&= \bar{\varepsilon} \left[-\dot{P}_N \theta^0 + \frac{1}{N} \left(N \dot{\theta}_0 + N^r \dot{\theta}_r \right) \theta^r (\theta^0)' - \frac{N^r}{N} \dot{\theta}_r \theta^r (\theta^0)' \right] = \bar{\varepsilon} \left[-\dot{P}_N \theta^0 + \dot{\theta}_0 \theta^r (\theta^0)' \right]; \tag{4.17}
\end{aligned}$$

$$\delta P_{N^r} = \bar{\varepsilon} \left[-\dot{P}_{N^r} \theta^0 + \dot{\theta}_r \theta^r (\theta^0)' \right]. \tag{4.18}$$

Here we also used the Hamiltonian equation $\dot{P}_N = -\frac{\delta H}{\delta N} = -\frac{\partial\mathcal{H}}{\partial N} + \left(\frac{\partial\mathcal{H}}{\partial N'} \right)'$ (B.2). Keeping in mind the relations between P_N, P_{N^r} and λ_0, λ_r (2.15), one can be convinced that Eqs. (4.9), (4.10) are correct.

One can find transformations for δP_V :

$$\begin{aligned}
\delta P_V &= \bar{\varepsilon} \left[-\dot{P}_V \theta^0 - P'_V \theta^r - P'_N \frac{\partial f}{\partial V} \theta^r - P'_{N^r} \frac{\partial f^r}{\partial V} \theta^r + P_N \frac{\partial^2 f}{\partial V^2} V (\theta^r)' + P_{N^r} \frac{\partial^2 f^r}{\partial V^2} V (\theta^r)' \right. \\
&\quad + P_N \frac{\partial^2 f}{\partial V^2} V N^r (\theta^0)' + P_{N^r} \frac{\partial^2 f^r}{\partial V^2} V N^r (\theta^0)' + \frac{\partial f^r}{\partial V} \bar{P}_{\theta^r} (\theta^0)' \theta^r \\
&\quad \left. + \frac{1}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^0)' \theta^r - \frac{N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^0)' \theta^r - \frac{N W W'}{V^2} (\theta^0)' \right].
\end{aligned} \tag{4.19}$$

The transformation of (2.20) gives the same result. Similarly one can obtain transformations in extended phase space for P_W and ghosts momenta. They are in correspondence with (2.21) – (2.25).

5. Concluding remarks

In the present paper we have constructed a self-consistent Hamiltonian dynamics for the generalized spherically symmetric model in extended phase space. Our starting point was the Faddeev – Popov effective action with gauge-fixing and ghost terms. Thanks to introducing the missing velocities into the Lagrangian by gauge conditions of special form we do not need to invent some prescription how to construct a Hamiltonian function.

Hamiltonian equations are proved to be equivalent to the Lagrangian set of equations. The group of transformations in extended phase space includes the group of gauge transformations for all gravitational degrees of freedom. We also have a clear algorithm how to construct a generator of transformations in extended phase space in accordance with the Noether theorem. The necessary condition for the algorithm to work is to find a BRST invariant form of the action. For the present model we have found the additional terms (4.13) that guarantees the required BRST invariance. The form of these terms gives us a hint what a BRST invariant form of the action would be in the full gravitational theory. Let us emphasize once again that we do not impose any additional conditions to ensure BRST invariance.

In our opinion, the proposed approach to construct Hamiltonian dynamics for gravity (and, in general, to any constrained system) is of interest by itself, as an alternative to the Dirac approach. On the other hand, it can be considered as a preliminary step to subsequent quantization of the model, and it will be a goal of our further research.

Appendix A. The extended set of Lagrangian equations

Variation of the effective action (2.16) with respect to N , N^r gives the constraints in the Lagrangian formalism which are equivalent to $\binom{0}{0}$ and $\binom{0}{1}$ Einstein equations:

$$\begin{aligned} \frac{\partial L}{\partial N} &= \partial_0 \frac{\partial L}{\partial \dot{N}} + \partial_r \frac{\partial L}{\partial \dot{N}^r}; \\ 0 &= \frac{\dot{V}W\dot{W}}{N^2} + \frac{V\dot{W}^2}{2N^2} + \frac{V}{2} - \frac{(W')^2}{2V} - \frac{WW''}{V} + \frac{V'W'W}{V^2} \\ &\quad - \frac{W'\dot{W}VN^r}{N^2} - \frac{W\dot{W}V(N^r)'}{N^2} - \frac{WW'\dot{V}N^r}{N^2} - \frac{W\dot{W}V'N^r}{N^2} \\ &\quad + \frac{(W')^2V(N^r)^2}{2N^2} + \frac{WW'V'(N^r)^2}{N^2} + \frac{WW'VN^r(N^r)'}{N^2} \\ &\quad + \dot{\pi}_N - \dot{\theta}_0\dot{\theta}^0 + (\dot{\theta}_0)'\theta^r + \dot{\theta}_0(\theta^r)' + \dot{\theta}_0N^r(\theta^0)' + \dot{\theta}_r\frac{2N}{V^2}(\theta^0)'; \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \frac{\partial L}{\partial N^r} &= \partial_0 \frac{\partial L}{\partial \dot{N}^r} + \partial_r \frac{\partial L}{\partial (N^r)'}; \\ 0 &= \frac{WW'N'VN^r}{N^2} - \frac{WW'\dot{V}}{N} - \frac{WW''VN^r}{N} + \frac{WV\dot{W}'}{N} - \frac{N'WV\dot{W}}{N^2} + \frac{WW'V'N^r}{N} \\ &\quad - \dot{\pi}_{N^r} + \dot{\theta}_r\dot{\theta}^0 - (\dot{\theta}_r)'\theta^r - 2\dot{\theta}_r(\theta^r)' - 2\dot{\theta}_rN^r(\theta^0)' - \dot{\theta}_0\frac{\partial f}{\partial V}V(\theta^0)' - \dot{\theta}_r\frac{\partial f^r}{\partial V}V(\theta^0)' - \dot{\theta}_0N(\theta^0)'. \end{aligned} \quad (\text{A.2})$$

Variations with respect to V , W leads to the equations which are equivalent to dynamical $\binom{1}{1}$ and $\binom{2}{2}$ Einstein equations:

$$\begin{aligned} \frac{\partial L}{\partial V} &= \partial_0 \frac{\partial L}{\partial \dot{V}} + \partial_r \frac{\partial L}{\partial \dot{V}^r}; \\ 0 &= \frac{\dot{W}^2}{2N} + \frac{W\ddot{W}}{N} - \frac{W\dot{W}\dot{N}}{N^2} + \frac{N}{2} - \frac{N(W')^2}{2V^2} - \frac{N'W'W}{V^2} \end{aligned}$$

$$\begin{aligned}
& - \frac{N'WW'(N^r)^2}{N^2} + \frac{WW''(N^r)^2}{N} - \frac{W'\dot{W}N^r}{N} + \frac{WW'N^r\dot{N}}{N^2} + \frac{N'N^rW\dot{W}}{N^2} \\
& - \frac{2W\dot{W}'N^r}{N} - \frac{WW'\dot{N}^r}{N} + \frac{(W')^2(N^r)^2}{2N} + \frac{WW'N^r(N^r)'}{N} \\
& - \dot{\pi}_N \frac{\partial f}{\partial V} - \dot{\pi}_{N^r} \frac{\partial f^r}{\partial V} - (\dot{\theta}_0)' \frac{\partial f}{\partial V} \theta^r + \dot{\theta}_0 \frac{\partial^2 f}{\partial V^2} V N^r (\theta^0)' \\
& + \dot{\theta}_0 \frac{\partial f}{\partial V} N^r (\theta^0)' + \dot{\theta}_0 \frac{\partial^2 f}{\partial V^2} V (\theta^r)' - \dot{\theta}_r \frac{2N^2}{V^3} (\theta^0)' \\
& + \dot{\theta}_r \frac{\partial^2 f^r}{\partial V^2} V N^r (\theta^0)' + \dot{\theta}_r \frac{\partial f^r}{\partial V} N^r (\theta^0)' + \dot{\theta}_r \frac{\partial^2 f^r}{\partial V^2} V (\theta^r)' - (\dot{\theta}_r)' \frac{\partial f^r}{\partial V} \theta^r; \tag{A.3}
\end{aligned}$$

$$\frac{\partial L}{\partial W} = \partial_0 \frac{\partial L}{\partial \dot{W}} + \partial_r \frac{\partial L}{\partial \dot{W}^r};$$

$$\begin{aligned}
0 = & \frac{W\ddot{V}}{N} + \frac{\dot{V}\dot{W}}{N} + \frac{V\ddot{W}}{N} - \frac{\dot{N}\dot{V}W}{N^2} - \frac{\dot{N}\dot{W}V}{N^2} - \frac{WN''}{V} + \frac{WN'V'}{V^2} - \frac{W'N'}{V} + \frac{W'NV'}{V^2} - \frac{W''N}{V} \\
& - \frac{2V\dot{W}'N^r}{N} - \frac{W'\dot{V}N^r}{N} - \frac{W'V\dot{N}^r}{N} + \frac{W'VN^r\dot{N}}{N^2} + \frac{\dot{W}VN^rN'}{N^2} - \frac{\dot{W}V'N^r}{N} - \frac{2W\dot{V}'N^r}{N} \\
& - \frac{WV'\dot{N}^r}{N} + \frac{WV'N^r\dot{N}}{N^2} - \frac{2W\dot{V}(N^r)'}{N} + \frac{W\dot{V}N^rN'}{N^2} - \frac{\dot{W}V(N^r)'}{N} - \frac{WV(\dot{N}^r)'}{N} + \frac{WV\dot{N}(N^r)'}{N^2} \\
& + \frac{WV''(N^r)^2}{N} + \frac{3WV'N^r(N^r)'}{N} - \frac{WV'(N^r)^2N'}{N^2} + \frac{WV((N^r)')^2}{N} + \frac{WV\dot{N}^r(N^r)''}{N} \\
& - \frac{WV\dot{N}^rN^r(N^r)'}{N^2} + \frac{W''V(N^r)^2}{N} + \frac{W'V'(N^r)^2}{N} + \frac{2W'V\dot{N}^r(N^r)'}{N} - \frac{W'V(N^r)^2N'}{N^2} \\
& - \dot{\pi}_N \frac{\partial f}{\partial W} - \dot{\pi}_{N^r} \frac{\partial f^r}{\partial W} - (\dot{\theta}_0)' \frac{\partial f}{\partial W} \theta^r - \dot{\theta}_0 \frac{\partial f}{\partial W} (\theta^r)' + \dot{\theta}_0 \frac{\partial^2 f}{\partial V \partial W} V N^r (\theta^0)' + \dot{\theta}_0 \frac{\partial^2 f}{\partial V \partial W} V (\theta^r)' \\
& + \dot{\theta}_r \frac{\partial^2 f^r}{\partial V \partial W} V N^r (\theta^0)' + \dot{\theta}_r \frac{\partial^2 f^r}{\partial V \partial W} V (\theta^r)' - (\dot{\theta}_r)' \frac{\partial f^r}{\partial W} \theta^r - \dot{\theta}_r \frac{\partial f^r}{\partial W} (\theta^r)'. \tag{A.4}
\end{aligned}$$

We also have four equations for two pairs of ghosts:

$$\frac{\partial L}{\partial \theta_0} = \partial_0 \frac{\partial L}{\partial \dot{\theta}_0} + \partial_r \frac{\partial L}{\partial (\dot{\theta}_0)'};$$

$$\begin{aligned}
0 = & N'\dot{\theta}^r - \frac{\partial f}{\partial V} V'\dot{\theta}^r - \frac{\partial f}{\partial W} W'\dot{\theta}^r + \dot{N}'\theta^r - \frac{\partial^2 f}{\partial V^2} \dot{V}V'\theta^r \\
& - \frac{\partial^2 f}{\partial V \partial W} V'\dot{W}\theta^r - \frac{\partial f}{\partial V} \dot{V}'\theta^r - \frac{\partial^2 f}{\partial V \partial W} \dot{V}W'\theta^r - \frac{\partial^2 f}{\partial W^2} \dot{W}W'\theta^r - \frac{\partial f}{\partial W} \dot{W}'\theta^r \\
& + \dot{N}\dot{\theta}^0 + N\ddot{\theta}^0 - \dot{N}N^r(\theta^0)' - N\dot{N}^r(\theta^0)' - NN^r(\dot{\theta}^0)' - \frac{\partial^2 f}{\partial V^2} \dot{V}VN^r(\theta^0)' \\
& - \frac{\partial^2 f}{\partial V \partial W} \dot{W}VN^r(\theta^0)' - \frac{\partial f}{\partial V} \dot{V}N^r(\theta^0)' - \frac{\partial f}{\partial V} V\dot{N}^r(\theta^0)' - \frac{\partial f}{\partial V} VN^r(\dot{\theta}^0)' \\
& - \frac{\partial^2 f}{\partial V^2} \dot{V}V(\theta^r)' - \frac{\partial^2 f}{\partial V \partial W} \dot{W}V(\theta^r)' - \frac{\partial f}{\partial V} \dot{V}(\theta^r)' - \frac{\partial f}{\partial V} V(\dot{\theta}^r)'; \tag{A.5}
\end{aligned}$$

$$\frac{\partial L}{\partial \theta^0} = \partial_0 \frac{\partial L}{\partial \dot{\theta}^0} + \partial_r \frac{\partial L}{\partial (\dot{\theta}^0)'};$$

$$\begin{aligned}
0 = & \ddot{\theta}_0 N + \dot{\theta}_0 \dot{N} - (\dot{\theta}_0)' NN^r - \dot{\theta}_0 N'N^r - \dot{\theta}_0 N(N^r)' \\
& - (\dot{\theta}_0)' \frac{\partial f}{\partial V} VN^r - \dot{\theta}_0 \frac{\partial^2 f}{\partial V^2} V'VN^r - \dot{\theta}_0 \frac{\partial^2 f}{\partial V \partial W} W'VN^r - \dot{\theta}_0 \frac{\partial f}{\partial V} V'N^r - \dot{\theta}_0 \frac{\partial f}{\partial V} V(N^r)'
\end{aligned}$$

$$\begin{aligned}
& + \ddot{\theta}_r N^r + \dot{\theta}_r \dot{N}^r - (\dot{\theta}_r)' \left(\frac{N^2}{V^2} + (N^r)^2 \right) - 2\dot{\theta}_r \frac{N N'}{V^2} + 2\dot{\theta}_r \frac{N^2 V'}{V^3} - 2\dot{\theta}_r N^r (N^r)' \\
& - (\dot{\theta}_r)' \frac{\partial f^r}{\partial V} V N^r - \dot{\theta}_r \frac{\partial^2 f^r}{\partial V^2} V' V N^r - \dot{\theta}_r \frac{\partial^2 f^r}{\partial V \partial W} W' V N^r - \dot{\theta}_r \frac{\partial f^r}{\partial V} V' N^r - \dot{\theta}_r \frac{\partial f^r}{\partial V} V (N^r)'; \quad (\text{A.6})
\end{aligned}$$

$$\frac{\partial L}{\partial \theta_r} = \partial_0 \frac{\partial L}{\partial \dot{\theta}_r} + \partial_r \frac{\partial L}{\partial (\theta_r)'};$$

$$\begin{aligned}
0 &= \dot{N}^r \dot{\theta}^0 + N^r \ddot{\theta}^0 - 2 \frac{N \dot{N}}{V^2} (\theta^0)' + 2 \frac{N^2 \dot{V}}{V^3} (\theta^0)' - \frac{N^2}{V^2} (\dot{\theta}^0)' - 2 N^r \dot{N}^r (\theta^0)' \\
&- (N^r)^2 (\dot{\theta}^0)' + \ddot{\theta}^r - \dot{N}^r (\theta^r)' - N^r (\dot{\theta}^r)' + (\dot{N}^r)' \theta^r + (N^r)' \dot{\theta}^r \\
&- \frac{\partial^2 f^r}{\partial V^2} \dot{V} V N^r (\theta^0)' - \frac{\partial^2 f^r}{\partial V^2} \dot{V} V (\theta^r)' - \frac{\partial^2 f^r}{\partial V^2} \dot{V} V' \theta^r \\
&- \frac{\partial^2 f^r}{\partial V \partial W} \dot{W} V N^r (\theta^0)' - \frac{\partial^2 f^r}{\partial V \partial W} \dot{W} V (\theta^r)' - \frac{\partial^2 f^r}{\partial V \partial W} \dot{W} V' \theta^r \\
&- \frac{\partial f^r}{\partial V} \dot{V} N^r (\theta^0)' - \frac{\partial f^r}{\partial V} V \dot{N}^r (\theta^0)' - \frac{\partial f^r}{\partial V} V N^r (\dot{\theta}^0)' - \frac{\partial f^r}{\partial V} \dot{V} (\theta^r)' - \frac{\partial f^r}{\partial V} V (\dot{\theta}^r)' \\
&- \frac{\partial f^r}{\partial V} \dot{V}' \theta^r - \frac{\partial f^r}{\partial V} V' \dot{\theta}^r - \frac{\partial^2 f^r}{\partial V \partial W} \dot{W} W' \theta^r - \frac{\partial^2 f^r}{\partial W^2} \dot{W} W' \theta^r - \frac{\partial f^r}{\partial W} \dot{W}' \theta^r - \frac{\partial f^r}{\partial W} W' \dot{\theta}^r; \quad (\text{A.7})
\end{aligned}$$

$$\frac{\partial L}{\partial \theta^r} = \partial_0 \frac{\partial L}{\partial \dot{\theta}^r} + \partial_r \frac{\partial L}{\partial (\theta^r)'};$$

$$\begin{aligned}
0 &= \dot{\theta}_0 N' - \dot{\theta}_0 \frac{\partial f}{\partial W} W' + (\dot{\theta}_0)' \frac{\partial f}{\partial V} V + \dot{\theta}_0 \frac{\partial^2 f}{\partial V^2} V' V + \dot{\theta}_0 \frac{\partial^2 f}{\partial V \partial W} W' V \\
&- \ddot{\theta}_r + (\dot{\theta}_r)' N^r + 2\dot{\theta}_r (N^r)' + (\dot{\theta}_r)' \frac{\partial f^r}{\partial V} V \\
&+ \dot{\theta}_r \frac{\partial^2 f^r}{\partial V^2} V' V + \dot{\theta}_r \frac{\partial^2 f^r}{\partial V \partial W} W' V - \dot{\theta}_r \frac{\partial f^r}{\partial W} W'. \quad (\text{A.8})
\end{aligned}$$

Variation with respect to π_N , π_{N^r} yields the gauge conditions (2.5), (2.6):

$$\dot{N} = \frac{\partial f}{\partial V} \dot{V} + \frac{\partial f}{\partial W} \dot{W}; \quad (\text{A.9})$$

$$\dot{N}^r = \frac{\partial f^r}{\partial V} \dot{V} + \frac{\partial f^r}{\partial W} \dot{W}. \quad (\text{A.10})$$

The equations (A.1) – (A.10) form the extended set of Lagrangian equations for the generalized spherically symmetric gravitational model.

Appendix B. The set of Hamiltonian equations in extended phase space

In this Appendix we present the full set of Hamiltonian equations in extended phase space.

$$\begin{aligned}
\dot{N} &= \frac{\partial f}{\partial V} V' N^r + \frac{\partial f}{\partial W} W' N^r + \frac{\partial f}{\partial V} V (N^r)' + \frac{N}{W} P_V \frac{\partial f}{\partial W} + \frac{N}{W} P_W \frac{\partial f}{\partial V} - \frac{N V}{W^2} P_V \frac{\partial f}{\partial V} \\
&- \frac{N V}{W^2} P_N \left(\frac{\partial f}{\partial V} \right)^2 + \frac{2N}{W} P_N \frac{\partial f}{\partial V} \frac{\partial f}{\partial W} + \frac{N}{W} P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial W} + \frac{N}{W} P_{N^r} \frac{\partial f}{\partial W} \frac{\partial f^r}{\partial V} - \frac{N V}{W^2} P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V}; \quad (\text{B.1}) \\
\dot{P}_N &= -\frac{1}{W} P_V P_W + \frac{V}{2W^2} P_V^2 + \frac{W'' W}{V} - \frac{V' W' W}{V^2} + \frac{(W')^2}{2V} - \frac{V}{2}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{W} P_V P_N \frac{\partial f}{\partial W} - \frac{1}{W} P_W P_N \frac{\partial f}{\partial V} + \frac{V}{W^2} P_V P_N \frac{\partial f}{\partial V} - \frac{1}{W} P_V P_{N^r} \frac{\partial f^r}{\partial W} - \frac{1}{W} P_W P_{N^r} \frac{\partial f^r}{\partial V} \\
& + \frac{V}{W^2} P_V P_{N^r} \frac{\partial f^r}{\partial V} + \frac{V}{2W^2} P_N^2 \left(\frac{\partial f}{\partial V} \right)^2 + \frac{V}{2W^2} P_{N^r}^2 \left(\frac{\partial f^r}{\partial V} \right)^2 - \frac{1}{W} P_N^2 \frac{\partial f}{\partial V} \frac{\partial f}{\partial W} - \frac{1}{W} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial f^r}{\partial W} \\
& - \frac{1}{W} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial W} - \frac{1}{W} P_N P_{N^r} \frac{\partial f}{\partial W} \frac{\partial f^r}{\partial V} + \frac{V}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V} \\
& + \frac{1}{N^2} \bar{P}_{\theta^0} P_{\bar{\theta}^0} - \frac{N^r}{N^2} \bar{P}_{\theta^r} P_{\bar{\theta}^0} - \frac{1}{N} (\bar{P}_{\theta^0})' \theta^r - \frac{1}{N} \bar{P}_{\theta^0} (\theta^r)' \\
& + \frac{(N^r)'}{N} \bar{P}_{\theta^r} \theta^r + \frac{N^r}{N} (\bar{P}_{\theta^r})' \theta^r + \frac{N^r}{N} \bar{P}_{\theta^r} (\theta^r)' - \frac{2N}{V^2} \bar{P}_{\theta^r} (\theta^0)' \\
& + \frac{V}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^r)' + \frac{V N^r}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^0)' - \frac{V N^r}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^r)' - \frac{V (N^r)^2}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^0)' \\
& + \frac{V'}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} \theta^r - \frac{V' N^r}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} \theta^r + \frac{W'}{N^2} \frac{\partial f}{\partial W} \bar{P}_{\theta^0} \theta^r - \frac{W' N^r}{N^2} \frac{\partial f}{\partial W} \bar{P}_{\theta^r} \theta^r; \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\dot{N}^r &= \frac{\partial f^r}{\partial V} V' N^r + \frac{\partial f^r}{\partial W} W' N^r + \frac{\partial f^r}{\partial V} V (N^r)' + \frac{N}{W} P_V \frac{\partial f^r}{\partial W} + \frac{N}{W} P_W \frac{\partial f^r}{\partial V} - \frac{NV}{W^2} P_V \frac{\partial f^r}{\partial V} \\
& - \frac{NV}{W^2} P_{N^r} \left(\frac{\partial f^r}{\partial V} \right)^2 + \frac{2N}{W} P_{N^r} \frac{\partial f^r}{\partial V} \frac{\partial f^r}{\partial W} + \frac{N}{W} P_N \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial W} \\
& + \frac{N}{W} P_N \frac{\partial f}{\partial W} \frac{\partial f^r}{\partial V} - \frac{NV}{W^2} P_N \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V}; \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
\dot{P}_{N^r} &= (P_V)' V - P_W W' - P_N \frac{\partial f}{\partial W} W' + (P_{N^r})' \frac{\partial f}{\partial V} V + P_N \frac{\partial^2 f}{\partial V^2} V V' + P_N \frac{\partial^2 f}{\partial V \partial W} V W' \\
& - P_{N^r} \frac{\partial f^r}{\partial W} W' + (P_{N^r})' \frac{\partial f^r}{\partial V} V + P_{N^r} \frac{\partial^2 f^r}{\partial V^2} V V' + P_{N^r} \frac{\partial^2 f^r}{\partial V \partial W} V W' \\
& + \frac{1}{N} \bar{P}_{\theta^r} P_{\bar{\theta}^0} - \bar{P}_{\theta^0} (\theta^0)' - 2 \bar{P}_{\theta^r} (\theta^r)' - (\bar{P}_{\theta^r})' \theta^r - \frac{N'}{N} \bar{P}_{\theta^r} \theta^r \\
& - \frac{\partial f^r}{\partial V} V \bar{P}_{\theta^r} (\theta^0)' - \frac{V}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} (\theta^0)' + \frac{V}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^r)' \\
& + \frac{2V N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} (\theta^0)' + \frac{V'}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} \theta^r + \frac{W'}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^r} \theta^r. \tag{B.4}
\end{aligned}$$

It can be checked by direct calculations that Eqs. (B.2), (B.4) coincide with the constraints equations in the Lagrangian formalism (A.1), (A.2), while Eqs. (B.1), (B.3) correspond to the gauge conditions (A.9), (A.10). The other Hamiltonian equations are:

$$\begin{aligned}
\dot{V} &= \frac{N}{W} P_W - \frac{NV}{W^2} P_V + V' N^r + V (N^r)' \\
& + \frac{N}{W} P_N \frac{\partial f}{\partial W} - \frac{NV}{W^2} P_N \frac{\partial f}{\partial V} + \frac{N}{W} P_{N^r} \frac{\partial f^r}{\partial W} - \frac{NV}{W^2} P_{N^r} \frac{\partial f^r}{\partial V}; \tag{B.5} \\
\dot{P}_V &= \frac{N}{2W^2} P_V^2 + (P_V)' N^r + \frac{N' W' W}{V^2} + \frac{N (W')^2}{2V^2} - \frac{N}{2} \\
& + (P_N)' \frac{\partial f}{\partial V} N^r - P_N \frac{\partial^2 f}{\partial V^2} V (N^r)' + (P_{N^r})' \frac{\partial f^r}{\partial V} N^r - P_{N^r} \frac{\partial^2 f^r}{\partial V^2} V (N^r)' \\
& - \frac{N}{W} P_V P_N \frac{\partial^2 f}{\partial V \partial W} - \frac{N}{W} P_W P_N \frac{\partial^2 f}{\partial V^2} + \frac{N}{W^2} P_V P_N \frac{\partial f}{\partial V} + \frac{NV}{W^2} P_V P_N \frac{\partial^2 f}{\partial V^2} \\
& - \frac{N}{W} P_V P_{N^r} \frac{\partial^2 f^r}{\partial V \partial W} - \frac{N}{W} P_W P_{N^r} \frac{\partial^2 f^r}{\partial V^2} + \frac{N}{W^2} P_V P_{N^r} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_V P_{N^r} \frac{\partial^2 f^r}{\partial V^2} \\
& + \frac{N}{2W^2} P_N^2 \left(\frac{\partial f}{\partial V} \right)^2 + \frac{NV}{W^2} P_N^2 \frac{\partial f}{\partial V} \frac{\partial^2 f}{\partial V^2} + \frac{N}{2W^2} P_{N^r}^2 \left(\frac{\partial f^r}{\partial V} \right)^2 + \frac{NV}{W^2} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial^2 f^r}{\partial V^2}
\end{aligned}$$

$$\begin{aligned}
& - \frac{N}{W} P_N^2 \frac{\partial^2 f}{\partial V^2} \frac{\partial f}{\partial W} - \frac{N}{W} P_N^2 \frac{\partial f}{\partial V} \frac{\partial^2 f}{\partial V \partial W} - \frac{N}{W} P_{N^r}^2 \frac{\partial^2 f^r}{\partial V^2} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial^2 f^r}{\partial V \partial W} \\
& - \frac{N}{W} P_N P_{N^r} \frac{\partial^2 f}{\partial V^2} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial^2 f^r}{\partial V \partial W} - \frac{N}{W} P_N P_{N^r} \frac{\partial^2 f}{\partial V \partial W} \frac{\partial f^r}{\partial V} - \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial W} \frac{\partial^2 f^r}{\partial V^2} \\
& + \frac{N}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_N P_{N^r} \frac{\partial^2 f}{\partial V^2} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial^2 f^r}{\partial V^2} \\
& + \frac{2N^2}{V^3} \bar{P}_{\theta^r}(\theta^0)' + \frac{\partial f^r}{\partial V} (\bar{P}_{\theta^r})' \theta^r - \frac{\partial^2 f^r}{\partial V^2} V N^r \bar{P}_{\theta^r}(\theta^0)' - \frac{\partial f^r}{\partial V} N^r \bar{P}_{\theta^r}(\theta^0)' - \frac{\partial^2 f^r}{\partial V^2} V \bar{P}_{\theta^r}(\theta^r)' \\
& - \frac{V}{N} \frac{\partial^2 f}{\partial V^2} \bar{P}_{\theta^0}(\theta^r)' - \frac{N^r}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^0}(\theta^0)' - \frac{V N^r}{N} \frac{\partial^2 f}{\partial V^2} \bar{P}_{\theta^0}(\theta^0)' + \frac{V N^r}{N} \frac{\partial^2 f}{\partial V^2} \bar{P}_{\theta^r}(\theta^r)' \\
& + \frac{(N^r)^2}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r}(\theta^0)' + \frac{V(N^r)^2}{N} \frac{\partial^2 f}{\partial V^2} \bar{P}_{\theta^r}(\theta^0)' - \frac{N'}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^0} \theta^r + \frac{1}{N} \frac{\partial f}{\partial V} (\bar{P}_{\theta^0})' \theta^r \\
& - \frac{(N^r)'}{N} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} \theta^r + \frac{N' N^r}{N^2} \frac{\partial f}{\partial V} \bar{P}_{\theta^r} \theta^r - \frac{N^r}{N} \frac{\partial f}{\partial V} (\bar{P}_{\theta^r})' \theta^r; \tag{B.6}
\end{aligned}$$

$$\dot{W} = \frac{N}{W} P_V + W' N^r + \frac{N}{W} P_N \frac{\partial f}{\partial V} + \frac{N}{W} P_{N^r} \frac{\partial f^r}{\partial V}; \tag{B.7}$$

$$\begin{aligned}
\dot{P}_W &= \frac{N}{W^2} P_V P_W - \frac{NV}{W^3} P_V^2 + (P_W)' N^r + P_W (N^r)' + \frac{N'' W}{V} - \frac{N' V' W}{V^2} + \frac{N' W'}{V} + \frac{N W''}{V} - \frac{N V' W'}{V^2} \\
&+ (P_N)' \frac{\partial f}{\partial W} N^r + P_N \frac{\partial f}{\partial W} (N^r)' - P_N \frac{\partial^2 f}{\partial V \partial W} V (N^r)' + (P_{N^r})' \frac{\partial f^r}{\partial W} N^r + P_{N^r} \frac{\partial f^r}{\partial W} (N^r)' \\
&- P_{N^r} \frac{\partial^2 f^r}{\partial V \partial W} V (N^r)' + \frac{N}{W^2} P_V P_N \frac{\partial f}{\partial W} - \frac{N}{W} P_V P_N \frac{\partial^2 f}{\partial W^2} + \frac{N}{W^2} P_W P_N \frac{\partial f}{\partial V} - \frac{N}{W} P_W P_N \frac{\partial^2 f}{\partial V \partial W} \\
&- \frac{2NV}{W^3} P_V P_N \frac{\partial f}{\partial V} + \frac{NV}{W^2} P_V P_N \frac{\partial^2 f}{\partial V \partial W} + \frac{N}{W^2} P_V P_{N^r} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_V P_{N^r} \frac{\partial^2 f^r}{\partial W^2} \\
&+ \frac{N}{W^2} P_W P_{N^r} \frac{\partial f^r}{\partial V} - \frac{N}{W} P_W P_{N^r} \frac{\partial^2 f^r}{\partial V \partial W} - \frac{2NV}{W^3} P_V P_{N^r} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_V P_{N^r} \frac{\partial^2 f^r}{\partial V \partial W} \\
&- \frac{NV}{W^3} P_N^2 \left(\frac{\partial f}{\partial V} \right)^2 + \frac{NV}{W^2} P_N^2 \frac{\partial f}{\partial V} \frac{\partial^2 f}{\partial V \partial W} - \frac{NV}{W^3} P_{N^r}^2 \left(\frac{\partial f^r}{\partial V} \right)^2 + \frac{NV}{W^2} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial^2 f^r}{\partial V \partial W} \\
&+ \frac{N}{W^2} P_N^2 \frac{\partial f}{\partial V} \frac{\partial f}{\partial W} - \frac{N}{W} P_N^2 \frac{\partial^2 f}{\partial V \partial W} \frac{\partial f}{\partial W} - \frac{N}{W} P_N^2 \frac{\partial f}{\partial V} \frac{\partial^2 f}{\partial W^2} \\
&+ \frac{N}{W^2} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_{N^r}^2 \frac{\partial^2 f^r}{\partial V \partial W} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_{N^r}^2 \frac{\partial f^r}{\partial V} \frac{\partial^2 f^r}{\partial W^2} \\
&+ \frac{N}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_N P_{N^r} \frac{\partial^2 f}{\partial V \partial W} \frac{\partial f^r}{\partial W} - \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial^2 f^r}{\partial W^2} \\
&+ \frac{N}{W^2} P_N P_{N^r} \frac{\partial f}{\partial W} \frac{\partial f^r}{\partial V} - \frac{N}{W} P_N P_{N^r} \frac{\partial^2 f}{\partial W^2} \frac{\partial f^r}{\partial V} - \frac{N}{W} P_N P_{N^r} \frac{\partial f}{\partial W} \frac{\partial^2 f^r}{\partial V \partial W} \\
&- \frac{2NV}{W^3} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_N P_{N^r} \frac{\partial^2 f}{\partial V \partial W} \frac{\partial f^r}{\partial V} + \frac{NV}{W^2} P_N P_{N^r} \frac{\partial f}{\partial V} \frac{\partial^2 f^r}{\partial V \partial W} \\
&+ \frac{\partial f^r}{\partial W} (\bar{P}_{\theta^r})' \theta^r + \frac{\partial f^r}{\partial W} \bar{P}_{\theta^r}(\theta^r)' - \frac{\partial^2 f^r}{\partial V \partial W} V N^r \bar{P}_{\theta^r}(\theta^0)' - \frac{\partial^2 f^r}{\partial V \partial W} V \bar{P}_{\theta^r}(\theta^r)' \\
&- \frac{V}{N} \frac{\partial^2 f}{\partial V \partial W} \bar{P}_{\theta^0}(\theta^r)' - \frac{V N^r}{N} \frac{\partial^2 f}{\partial V \partial W} \bar{P}_{\theta^0}(\theta^0)' + \frac{V N^r}{N} \frac{\partial^2 f}{\partial V \partial W} \bar{P}_{\theta^r}(\theta^r)' \\
&+ \frac{V(N^r)^2}{N} \frac{\partial^2 f}{\partial V \partial W} \bar{P}_{\theta^r}(\theta^0)' - \frac{N'}{N^2} \frac{\partial f}{\partial W} \bar{P}_{\theta^0} \theta^r + \frac{1}{N} \frac{\partial f}{\partial W} (\bar{P}_{\theta^0})' \theta^r + \frac{1}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^0}(\theta^r)' \\
&- \frac{(N^r)'}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^r} \theta^r + \frac{N^r N'}{N^2} \frac{\partial f}{\partial W} \bar{P}_{\theta^r} \theta^r - \frac{N^r}{N} \frac{\partial f}{\partial W} (\bar{P}_{\theta^r})' \theta^r - \frac{N^r}{N} \frac{\partial f}{\partial W} \bar{P}_{\theta^r}(\theta^r)'. \tag{B.8}
\end{aligned}$$

Eqs. (B.5) – (B.8) are equivalent to the dynamical Lagrangian equations (A.3), (A.4).

$$\dot{\theta}^0 = \frac{1}{N} P_{\theta^0} - \frac{N'}{N} \theta^r + N^r (\theta^0)' + \frac{V}{N} \frac{\partial f}{\partial V} (\theta^r)' + \frac{V N^r}{N} \frac{\partial f}{\partial V} (\theta^0)' + \frac{V'}{N} \frac{\partial f}{\partial V} \theta^r + \frac{W'}{N} \frac{\partial f}{\partial W} \theta^r; \tag{B.9}$$

$$\dot{P}_{\theta_0} = 0; \quad (B.10)$$

$$\dot{\bar{\theta}}_0 = \frac{1}{N}\bar{P}_{\theta^0} - \frac{N^r}{N}\bar{P}_{\theta^r}; \quad (B.11)$$

$$\begin{aligned} \dot{P}_{\theta^0} &= (N^r)'\bar{P}_{\theta^0} + N^r(\bar{P}_{\theta^0})' + \frac{2NN'}{V^2}\bar{P}_{\theta^r} - \frac{2N^2V'}{V^3}\bar{P}_{\theta^r} + \frac{N^2}{V^2}(\bar{P}_{\theta^r})' \\ &+ \frac{\partial^2 f^r}{\partial V^2}VV'N^r\bar{P}_{\theta^r} + \frac{\partial^2 f^r}{\partial V\partial W}VW'N^r\bar{P}_{\theta^r} + \frac{\partial f^r}{\partial V}V'N^r\bar{P}_{\theta^r} + \frac{\partial f^r}{\partial V}V(N^r)'\bar{P}_{\theta^r} + \frac{\partial f^r}{\partial V}VN^r(\bar{P}_{\theta^r})' \\ &+ \frac{V'N^r}{N}\frac{\partial f}{\partial V}\bar{P}_{\theta^0} + \frac{V(N^r)'}{N}\frac{\partial f}{\partial V}\bar{P}_{\theta^0} - \frac{VN'N^r}{N^2}\frac{\partial f}{\partial V}\bar{P}_{\theta^0} + \frac{VV'N^r}{N}\frac{\partial^2 f}{\partial V^2}\bar{P}_{\theta^0} + \frac{VW'N^r}{N}\frac{\partial^2 f}{\partial V\partial W}\bar{P}_{\theta^0} \\ &+ \frac{VN^r}{N}\frac{\partial f}{\partial V}(\bar{P}_{\theta^0})' - \frac{V'(N^r)^2}{N}\frac{\partial f}{\partial V}\bar{P}_{\theta^r} - \frac{2VN^r(N^r)'}{N}\frac{\partial f}{\partial V}\bar{P}_{\theta^r} + \frac{VN'(N^r)^2}{N^2}\frac{\partial f}{\partial V}\bar{P}_{\theta^r} \\ &- \frac{VV'(N^r)^2}{N}\frac{\partial^2 f}{\partial V^2}\bar{P}_{\theta^r} - \frac{VW'(N^r)^2}{N}\frac{\partial^2 f}{\partial V\partial W}\bar{P}_{\theta^r} - \frac{V(N^r)^2}{N}\frac{\partial f}{\partial V}(\bar{P}_{\theta^r})'; \end{aligned} \quad (B.12)$$

$$\begin{aligned} \dot{\theta}^r &= P_{\theta^r} - \frac{N^r}{N}P_{\bar{\theta}^0} + N^r(\theta^r)' - (N^r)'\theta^r + \frac{N'N^r}{N}\theta^r + \frac{N^2}{V^2}(\theta^0)' \\ &+ \frac{\partial f^r}{\partial V}V'\theta^r + \frac{\partial f^r}{\partial W}W'\theta^r + \frac{\partial f^r}{\partial V}VN^r(\theta^0)' + \frac{\partial f^r}{\partial V}V(\theta^r)' \\ &- \frac{VN^r}{N}\frac{\partial f}{\partial V}(\theta^r)' - \frac{V(N^r)^2}{N}\frac{\partial f}{\partial V}(\theta^0)' - \frac{V'N^r}{N}\frac{\partial f}{\partial V}\theta^r - \frac{W'N^r}{N}\frac{\partial f}{\partial W}\theta^r; \end{aligned} \quad (B.13)$$

$$\dot{P}_{\theta^r} = 0; \quad (B.14)$$

$$\dot{\bar{\theta}}_r = \bar{P}_{\theta^r}; \quad (B.15)$$

$$\begin{aligned} \dot{P}_{\theta^r} &= \frac{N'}{N}\bar{P}_{\theta^0} + 2(N^r)'\bar{P}_{\theta^r} + N^r(\bar{P}_{\theta^r})' - \frac{N'N^r}{N}\bar{P}_{\theta^r} - \frac{\partial f^r}{\partial W}W'\bar{P}_{\theta^r} \\ &+ \frac{\partial^2 f^r}{\partial V^2}VV'\bar{P}_{\theta^r} + \frac{\partial^2 f^r}{\partial V\partial W}VW'\bar{P}_{\theta^r} + \frac{\partial f^r}{\partial V}V(\bar{P}_{\theta^r})' - \frac{N'V}{N^2}\frac{\partial f}{\partial V}\bar{P}_{\theta^0} \\ &+ \frac{VV'}{N}\frac{\partial^2 f}{\partial V^2}\bar{P}_{\theta^0} + \frac{VW'}{N}\frac{\partial^2 f}{\partial V\partial W}\bar{P}_{\theta^0} + \frac{V}{N}\frac{\partial f}{\partial V}(\bar{P}_{\theta^0})' - \frac{V(N^r)'}{N}\frac{\partial f}{\partial V}\bar{P}_{\theta^r} \\ &+ \frac{VN'N^r}{N^2}\frac{\partial f}{\partial V}\bar{P}_{\theta^r} - \frac{VV'N^r}{N}\frac{\partial^2 f}{\partial V^2}\bar{P}_{\theta^r} - \frac{VW'N^r}{N}\frac{\partial^2 f}{\partial V\partial W}\bar{P}_{\theta^r} - \frac{VN^r}{N}\frac{\partial f}{\partial V}(\bar{P}_{\theta^r})' \\ &- \frac{W'}{N}\frac{\partial f}{\partial W}\bar{P}_{\theta^0} + \frac{W'N^r}{N}\frac{\partial f}{\partial W}\bar{P}_{\theta^r}. \end{aligned} \quad (B.16)$$

Eqs. (B.9) – (B.16) are equivalent to the equations for ghosts (A.5) – (A.8).

Appendix C. The explicit expression for addition terms in the gauged Einstein equations

In this Appendix the additional terms in the Einstein equation (2.17), $T_{\mu(obs)}^\nu$ and $T_{\mu(ghost)}^\nu$, which can be obtained by varying the gauge-fixing and ghost action, respectively, are given in the explicit form.

$$T_{0(obs)}^0 = \frac{1}{4\pi} \left(\dot{\pi}_N \frac{1}{VW^2} + \dot{\pi}_{N^r} \frac{N^r}{NVW^2} \right); \quad (C.1)$$

$$T_{1(obs)}^0 = \frac{1}{4\pi} \dot{\pi}_{N^r} \frac{1}{NVW^2}; \quad (C.2)$$

$$T_{1(obs)}^1 = -\frac{1}{4\pi} \left[\dot{\pi}_N \frac{1}{NW^2} \frac{\partial f}{\partial V} + \dot{\pi}_{N^r} \left(\frac{N^r}{NVW^2} + \frac{1}{NW^2} \frac{\partial f^r}{\partial V} \right) \right]; \quad (C.3)$$

$$T_{2(obs)}^2 = T_{3(obs)}^3 = -\frac{1}{8\pi} \left(\dot{\pi}_N \frac{1}{NVW} \frac{\partial f}{\partial W} + \dot{\pi}_{N^r} \frac{1}{NVW} \frac{\partial f^r}{\partial W} \right). \quad (C.4)$$

$$\begin{aligned}
T_{0(ghost)}^0 &= \frac{1}{4\pi} \left[-\frac{1}{VW^2} \dot{\theta}_0 \dot{\theta}^0 + \frac{1}{VW^2} \left(\dot{\theta}_0 \theta^r \right)' + \frac{2N}{V^3 W^2} \dot{\theta}_r (\theta^0)' + \frac{2N^r}{VW^2} \dot{\theta}_0 (\theta^0)' \right. \\
&+ \frac{N^r}{NVW^2} \left(\dot{\theta}_r \right)' \theta^r + \frac{2N^r}{NVW^2} \dot{\theta}_r (\theta^r)' - \frac{N^r}{NVW^2} \dot{\theta}_r \dot{\theta}^0 + \frac{2(N^r)^2}{NVW^2} \dot{\theta}_r (\theta^0)' \\
&\left. + \frac{N^r}{NW^2} \frac{\partial f}{\partial V} \dot{\theta}_0 (\theta^0)' + \frac{N^r}{NW^2} \frac{\partial f^r}{\partial V} \dot{\theta}_r (\theta^0)' \right]; \tag{C.5}
\end{aligned}$$

$$\begin{aligned}
T_{1(ghost)}^0 &= \frac{1}{4\pi} \left[\frac{1}{VW^2} \dot{\theta}_0 (\theta^0)' + \frac{1}{NVW^2} \left(\dot{\theta}_r \right)' \theta^r + \frac{2}{NVW^2} \dot{\theta}_r (\theta^r)' - \frac{1}{NVW^2} \dot{\theta}_r \dot{\theta}^0 \right. \\
&\left. + \frac{2N^r}{NVW^2} \dot{\theta}_r (\theta^0)' + \frac{1}{NW^2} \frac{\partial f}{\partial V} \dot{\theta}_0 (\theta^0)' + \frac{1}{NW^2} \frac{\partial f^r}{\partial V} \dot{\theta}_r (\theta^0)' \right]; \tag{C.6}
\end{aligned}$$

$$\begin{aligned}
T_{1(ghost)}^1 &= \frac{1}{4\pi} \left[-\frac{2N}{V^3 W^2} \dot{\theta}_r (\theta^0)' - \frac{N^r}{NVW^2} \left(\dot{\theta}_r \right)' \theta^r - \frac{N^r}{VW^2} \dot{\theta}_0 (\theta^0)' - \frac{2N^r}{NVW^2} \dot{\theta}_r (\theta^r)' \right. \\
&+ \frac{N^r}{NVW^2} \dot{\theta}_r \dot{\theta}^0 - \frac{2(N^r)^2}{NVW^2} \dot{\theta}_r (\theta^0)' - \frac{1}{NW^2} \frac{\partial f}{\partial V} \left(\dot{\theta}_0 \right)' \theta^r \\
&- \frac{1}{NW^2} \frac{\partial f^r}{\partial V} \left(\dot{\theta}_r \right)' \theta^r + \frac{V}{NW^2} \frac{\partial^2 f}{\partial V^2} \dot{\theta}_0 (\theta^r)' + \frac{V}{NW^2} \frac{\partial^2 f^r}{\partial V^2} \dot{\theta}_r (\theta^r)' \\
&\left. + \frac{VN^r}{NW^2} \frac{\partial^2 f}{\partial V^2} \dot{\theta}_0 (\theta^0)' + \frac{VN^r}{NW^2} \frac{\partial^2 f^r}{\partial V^2} \dot{\theta}_r (\theta^0)' \right]; \tag{C.7}
\end{aligned}$$

$$\begin{aligned}
T_{2(ghost)}^2 = T_{3(ghost)}^3 &= -\frac{1}{8\pi} \left[\frac{1}{NVW} \frac{\partial f}{\partial W} \left(\dot{\theta}_0 \right)' \theta^r + \frac{1}{NVW} \frac{\partial f^r}{\partial W} \left(\dot{\theta}_r \right)' \theta^r + \frac{1}{NVW} \frac{\partial f}{\partial W} \dot{\theta}_0 (\theta^r)' \right. \\
&+ \frac{1}{NVW} \frac{\partial f^r}{\partial W} \dot{\theta}_r (\theta^r)' - \frac{N^r}{NW} \frac{\partial^2 f}{\partial V \partial W} \dot{\theta}_0 (\theta^0)' - \frac{N^r}{NW} \frac{\partial^2 f^r}{\partial V \partial W} \dot{\theta}_r (\theta^0)' \\
&\left. - \frac{1}{NW} \frac{\partial^2 f}{\partial V \partial W} \dot{\theta}_0 (\theta^r)' - \frac{1}{NW} \frac{\partial^2 f^r}{\partial V \partial W} \dot{\theta}_r (\theta^r)' \right]. \tag{C.8}
\end{aligned}$$

The rest components are equal to zero. When comparing (C.1) – (C.4) and (C.5) – (C.8) with the terms in (A.1) – (A.4) that include π_N , π_{N^r} and ghost variables, one should keep in mind that Eq. (A.1) is a linear combination of $\binom{0}{0}$ and $\binom{0}{1}$ Einstein equations, namely,

$$\frac{N^2 VW^2}{2} (g^{00} G_0^0 + g^{01} G_1^0),$$

where $G_\mu^\nu = R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R$ is the left-hand side of the Einstein equations (2.17), Eq. (A.2) is the $\binom{0}{1}$ Einstein equation multiplied by $\frac{NVW^2}{2}$, Eq. (A.3) is a linear combination of $\binom{0}{1}$ and $\binom{1}{1}$ Einstein equations,

$$-\frac{NW^2}{2V^2} (g_{10} G_1^0 + g_{11} G_1^1),$$

and Eq. (A.4) is the $\binom{2}{2}$ (or $\binom{3}{3}$) Einstein equation multiplied by $(-NVW)$.

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